COMBINATORICS OF TWO SECOND ORDER MOCK THETA FUNCTIONS

HANNAH BURSON

1. Introduction

In January 1920, Ramanujan wrote his final letter to Hardy [3], where he introduced a class of functions that he called mock theta functions due to how they mimic certain behaviors of classical modular forms, especially Jacobi theta functions. In the letter, Ramanujan listed properties of 17 such functions. Based on some of these properties, he assigned an order to each mock theta function, classifying his examples as third, fifth, or seventh order mock theta functions.

In the century since Ramanujan's death, number theorists have investigated the properties of these functions and worked to classify exactly how "almost modular" these functions are. This rigorous study started with Watson [12], who proved many of Ramanujan's initial claims and introduced some new mock theta functions. However, it was not until Zwegers's 2002 thesis [13] that we began to understand how mock theta functions fit in the general area of non-holomorphic modular forms. Zwegers showed that mock theta functions are, in fact, not modular, but can be completed to become a function that displays modular behavior by adding a certain non-holomorphic part. This completed function is called a weak Maass form. For more information on the history and definition of mock theta functions, see [2], [7],[9], or [11].

Zwegers's work allowed for the discovery of infinite families of mock theta functions. In particular, McIntosh studied some second order mock theta functions in [10]. In his study, McIntosh stated the following second order mock theta function identities:

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{1}$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{2}$$

We use the q-shifted factorial

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

In this paper, we assume $|q|<1$. Then, we can write

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k).$$

The main purpose of this paper is to provide combinatorial proofs of (1) and (2).

In Section 3, we will prove a general identity that contains (1) and (2) as special cases. In Section 4, we will discuss other special cases of the combinatorial interpretation from Section 3, including a more direct combinatorial interpretation of (1). Finally, in Section 5, we generalize our interpretation of (2) to give a combinatorial interpretation of a q-hypergeometric series transformation identity.

2. Background

A partition λ of an integer n is a multiset of integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We use the convention that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and let $p(n)$ denote the number of partitions of n. A generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Partitions have a graphical representation called a Ferrers diagram (or Young diagram), which is a left-justified array of cells where the jth row has λ_j cells.

In this paper, we will use two generalizations of partitions. A k-colored partition is a partition in which each part can appear in one of k colors. Let $p_k(n)$ denote the number of k-colored partitions of n. Then we have the following generating function:

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty^k}.$$

In [1], George Andrews introduced odd Ferrers diagrams, which are obtained by taking a Ferrers diagram, placing a 0 in the top-left corner, filling the top row and left column with 1's, and filling the remainder of the diagram with 2's. We can think of these diagrams as representing a pair (k, λ) where k is a non-negative integer and λ is a partition into odd parts of size at most $2k + 1$.

Overpartitions [5] are partitions where the first appearance of any size part may be overlined. If we let $\overline{p}(n)$ be the function that counts the number of overpartitions of n, we can see that a generating function for $\overline{p}(n)$ is given by

$$\frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Overpartitions are a logical generalization of partitions and can be extended to provide a generalization of many partition-theoretic objects. In this paper we will work with an overpartition generalization of odd Ferrers diagrams called shaded odd Ferrers diagrams. In this generalization, we may add a shaded box to any interior corner of an odd Ferrers diagram. This is equivalent to considering pairs $(k, \overline{\lambda})$ where k is a non-negative integer and λ is an overpartition into odd parts of size at most $2k+1$ where all overlined parts must be of size at most $2k - 1$. Shaded odd Ferrers diagrams are equivalent to the boxed 2-modular diagrams introduced by the author in [4]. We define the size of a shaded odd Ferrers diagram to be the sum of the entries in all the cells. If we let $\mathrm{so}(n)$ denote the number of shaded odd Ferrers diagrams of size n, we have the following generating function:

$$\sum_{n=0}^{\infty} \mathrm{so}(n)q^n = \sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q;q^2)_{k+1}} q^k.$$

(a) an odd Ferrers diagram (b) a shaded odd Ferrers diagram

Figure 1

Note that the sum on the right-hand side is indexed over the number of ones in the top row.

3. A general identity

In this section, we will focus on proving the following theorem.

Theorem 3.1. *For $|q| < 1$, $|atz| \leq 1$ and $z \neq 0$,*

$$\sum_{n=0}^{\infty} \frac{(\frac{-q^2}{z}; q^2)_n (a^2tz)^n q^{n(n+1)}}{(atq; q^2)_{n+1}(aq; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(at)^n q^n (-azq; q^2)_n}{(aq; q^2)_{n+1}}.$$

Proof. This identity can be obtained by a ${}_3\phi_2$ transformation (c.f. [6, Eq. 17.9.6]), which states that

$$\frac{(e/a, de/(bc); q)_\infty}{(e, de/(abc); q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, d/b, d/c \\ d, de/(bc) \end{matrix} ; q, e/a \right) = {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; q, de/(abc) \right), \tag{3}$$

where

$$(a, b; q)_\infty = (a; q)_\infty (b; q)_\infty$$

and

$${}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n (c; q)_n}{(d; q)_n (e; q)_n (q; q)_n} z^n, \quad |z| < 1.$$

In (3), replacing q by q^2 and taking $a = q^2$, $b = \frac{-q^2}{\tau}$, $c = \frac{-q^2}{z}$, $d = aq^3$ and $e = atq^3$ gives

$$\sum_{n=0}^{\infty} \frac{(-a\tau q; q^2)_n (-azq; q^2)_n}{(aq; q^2)_{n+1}(a^2t\tau z; q^2)_{n+1}} a^n t^n q^n = \sum_{n=0}^{\infty} \frac{(-q^2/\tau; q^2)_n (-q^2/z; q^2)_n}{(aq; q^2)_{n+1}(atq; q^2)_{n+1}} a^{2n} t^n \tau^n z^n.$$

Then, by letting $\tau \to 0$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-azq; q^2)_n}{(aq; q^2)_{n+1}} a^n t^n q^n &= \lim_{\tau \to 0} \frac{1}{(1-atq)(1-aq)} \sum_{n=0}^{\infty} \frac{(-q^2/\tau; q^2)_n (-q^2/z; q^2)_n}{(atq^3; q^2)_n (aq^3; q^2)_n} a^{2n} t^n \tau^n z^n \\ &= \sum_{n=0}^{\infty} \frac{(-q^2/z; q^2)_n a^{2n} t^n z^n q^{n(n+1)}}{(atq; q^2)_{n+1}(aq; q^2)_{n+1}}. \end{aligned} \tag{4}$$

□

In order to give a combinatorial proof of Theorem 3.1, we introduce some notation. Let $\mathcal{P}_{B,k}$ be the set of two-colored partitions into parts of size $\leq 2k+1$, such that all even part sizes $\leq 2k$ are color 2 and appear exactly one or two times. Because all even parts are the same color, we will not label the color of the even parts. We let $\mathcal{P}_B = \cup_{k\geq 0}\mathcal{P}_{B,k}$. Let $\mathcal{SF}$ be the set of shaded odd Ferrers diagrams.

Given any partition λ, we let $\nu(\lambda)$ denote the number of parts and $\nu_o(\lambda)$ denote the number of odd parts. Analogously, for a shaded odd Ferrers diagram π, $\nu(\pi)$ is the number of rows in the diagram. For a partition $\lambda \in \mathcal{P}_B$, we define $\nu_{d,e}$ to be the number of even part sizes that appear, $\nu_{u,e}(\lambda)$ to be the number of even parts that appear exactly once and $\nu_1(\pi)$ to be the number of odd parts of color 1. For example, if we take $\lambda = 7_1+7_2+6+6+5_2+5_2+4+3_1+2+2+1_1+1_1+1_2 \in \mathcal{P}_{B,3}$, we have $\nu(\lambda)=13$, $\nu_o(\lambda)=8$, $\nu_{d,e}(\lambda)=3$, $\nu_{u,e}(\lambda)=1$, and $\nu_1(\pi)=4$. Note that if $\lambda \in \mathcal{P}_{B,k}$, then $\nu_{d,e}=k$. For a shaded odd Ferrers diagram π, we define $\overline{\nu}(\pi)$ to be the number of shaded cells in π and $\text{top}(\pi)$ to be the sum of the entries in the top row of π. Additionally, when we talk about row numbers of a shaded odd Ferrers diagram, we label the rows starting with the top row as row 0. Figure 2 illustrates these statistics for a shaded odd Ferrers diagram.

row 0	0	1	1	1	1	1
row 1	1	2	2	2	2	2
row 2	1	2	2	2		
row 3	1	2	2	2		
row 4	1	2	2			
row 5	1	2	2			
row 6	1					

FIGURE 2. a shaded odd Ferrers diagram π with $\nu(\pi)=7$, $\overline{\nu}(\pi)=2$, and $\text{top}(\pi)=5$

Combinatorial proof of Theorem 3.1. First, notice that

$$\sum_{k=0}^{\infty} \frac{(\frac{-q^2}{z};q^2)_k (a^2tz)^k q^{k(k+1)}}{(atq;q^2)_{k+1}(aq;q^2)_{k+1}} = \sum_{\lambda\in\mathcal{P}} a^{\nu_o(\lambda)+2\nu_{d,e}(\lambda)} t^{\nu_{d,e}(\lambda)+\nu_1(\lambda)} z^{\nu_{u,e}(\lambda)} q^{|\lambda|}$$

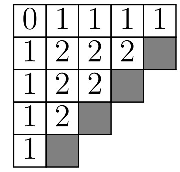
and

$$\sum_{n=0}^{\infty} \frac{(at)^n q^n(-azq;q^2)_n}{(aq;q^2)_{n+1}} = \sum_{\pi\in\mathcal{SF}} a^{\text{top}(\pi)+\nu(\pi)-1} t^{\text{top}(\pi)} z^{\overline{\nu}(\pi)} q^{|\pi|}.$$

To prove Theorem 3.1 combinatorially, we will construct a bijection ϕ between $\mathcal{P}_B$ and $\mathcal{SF}$. Let $k \geq 0$ and let $\lambda \in \mathcal{P}_{B,k}$. We define $\phi(\lambda)$ by creating a shaded odd Ferrers diagram according to the following procedure (the column on the right will illustrate the procedure on the partition $9_1+9_1+8+7_1+7_2+7_2+6+6+4+3_1+2+2+1_2 \in \mathcal{P}_{B,4}$):

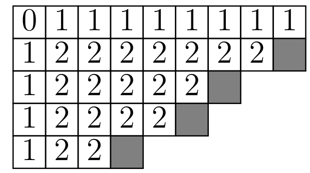
Step 1. For each $1 \le j \le k$, turn the first part of size $2j$ into one shaded row of size $2j-1$ and add a single 1 to the top row of the diagram. After this step, there will be k 1s in the top row and k shaded rows (sizes $\{1, 3, \dots, 2k-1\}$). Represent the result as a shaded odd Ferrers diagram.

$8 + 6 + 4 + 2 \;\to$



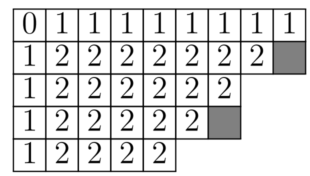
Step 2. Take each part of color 1, and add it as a column to the boxed 2-modular diagram.

$+\,9_1 + 9_1$
$+\,7_1 + 3_1 \;\to$



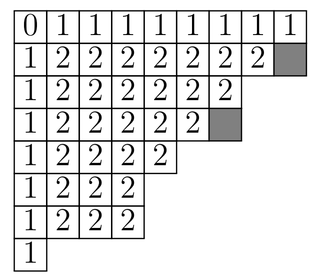
Step 3. Let $\{\tau_1, \tau_2, \dots, \tau_m\}$ be the remaining even parts, listed in decreasing order. For each $1 \le j \le m$ define $g(j)$ to be the maximum integer n such that there are at least n parts of color 1 and size $\ge \tau_j + (2(j+n)-1)$. Then, remove the shaded box from row $j + g(j)$ and append τ_j as a 2-modular column starting at row $j + g(j)$.

$+\,6 + 2 \;\to$



Step 4. Add parts of color 2 as unshaded 2-modular rows.

$+\,7_2 + 7_2 + 1_2 \;\to$



Note that, by the definitions in Step 3, for $1 \le j < m$, λ has at least $g(j)$ parts of color 1 and size at least

$$\begin{aligned}\tau_j + 2(j+g(j)) - 1 &\ge \tau_{j+1} + 2 + 2(j+g(j)) - 1 \quad (\text{because } \tau_{j+1} \le \tau_j - 2)\\ &= \tau_{j+1} + 2(j+1+g(j)) - 1.\end{aligned}$$

Therefore, $g(j+1) \ge g(j)$. Thus $j+1+g(j+1) > j+g(j)$. This inequality shows that Step 3 is well-defined.

To help us define the inverse map, we prove the following claim.
Claim: Using the definitions in Step 3, for $1 \le j < m$, we have

$$\tau_{j+1} + 2g(j+1) + 2j + 2 \le \tau_j + 2g(j) + 2j. \tag{5}$$

Proof of Claim. Assume that the claim is false. Since $\tau_{j+1} < \tau_j$, we can define $x > 0$ such that $\tau_{j+1} = \tau_j - 2x$. Then, by our assumption, we have

$$2g(j+1) - 2x + 2 > 2g(j).$$

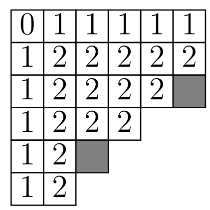
Then, by definition, λ has at least $g(j+1)$ parts of color 1 and size at least

$$\tau_j - 2x + 2(j+1+g(j+1)) - 1 \geq \tau_j + 2j + 2 + 2g(j) - 1.$$

Note that our assumption implies that $g(j+1) > g(j)$, so we now can say that there are at least $g(j)+1$ parts of color 1 and size at least $\tau_j + 2(j+g(j)+1) - 1$. This contradicts the assumption that $g(j)$ is the maximum integer n such that there are at least n parts of color 1 and size $\geq \tau_j + (2(j+n)-1)$. Therefore, our assumption must be false and our claim must be true. □

To prove that ϕ is a bijection, we give the inverse map ϕ^{-1}. We create a partition in $\bigcup_{k\geq 0} \mathcal{P}_{B,k}$ from each shaded odd Ferrers diagram according to the following procedure:

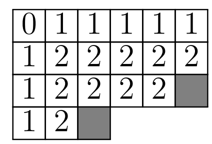
Step 1. Let $\pi \in \mathcal{SF}$. For $j \geq 1$, let n_j be the size of the jth largest non-shaded row (not including the top row). Create λ_2, a partition into odd parts by taking all n_j such that $n_j < 2\overline{\nu}(\pi) + 2j + 1$. Let π' be the rows of π that were not used to create λ_2.



$\rightarrow$ $\overline{\nu}(\pi) = 2,$
$n_1 = 11 \not< 7,$
$n_2 = 7 < 9,$
$n_3 = 3 < 11,$
$\pi_2 = 7 + 3$

Step 2. For each $j \geq 1$, starting at 1, if row j of π' is not shaded, create an even part by removing one cell containing a 2 from π'_j and one cell containing a 2 from each row π'_{j+m} with $\text{size}(\pi'_{j+m}) > 2(\nu - m)$, where $\nu = \nu(\pi')$ is the number of rows in π'. Then, add a shaded box to the end of row j. Let λ'_e be the collection of the even parts obtained in this step, noting that λ'_e will have distinct parts.

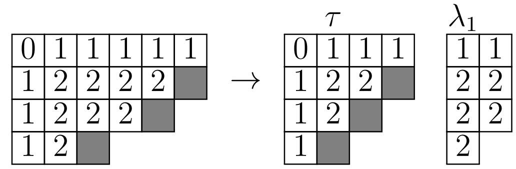
π' :



$\rightarrow$ $\nu(\pi') = 4$
$\text{size}(\pi'_3) = 9 > 2 + 6$
$\text{size}(\pi'_4) = 3 \not> 0 + 4$
$\lambda'_e = 4$

Step 3. Note that, after Step 2, the remaining diagram will have distinct parts. Thus, we can separate it into an $\nu \times \nu$ Durfee triangle odd Ferrers diagram (called τ) and the conjugate of a 2-modular diagram for a partition into odd parts of size $\leq 2\nu + 1$ (called λ_1).

τ λ_1



Step 4. From the triangle, create the even parts $2 + 4 + \ldots + 2\nu$. Inserting the parts of λ'_e, the parts of λ_1 as parts of color 1, and the parts of λ_2 as parts of color 2, we obtain a partition in $\mathcal{P}_B$.

$\tau \rightarrow 6 + 4 + 2$
$\lambda_1 \rightarrow 7_1 + 5_1$
$\lambda'_e = 4$
$\lambda_2 \rightarrow 7_2 + 3_2$

$\rightarrow$ $\phi^{-1}(\pi)$:
$7_1 + 7_2 + 6 + 5_1 + 4 + 4 + 3_2 + 2$

To confirm that ϕ^{-1} is, in fact, the inverse of ϕ, we show how the threshold conditions are related. Let $\lambda \in \mathcal{P}_B$ and define $\pi = \phi(\lambda) \in \mathcal{SF}$. Note that any unshaded row in π that comes from an odd part of color 2 in λ must have size at most $2\nu_{d,e}(\lambda)+1$. Then, using (5), we can see that, once we remove the parts that came from an odd part of color 2, any remaining unshaded rows must satisfy $|\pi_{j+g(j)}| \geq 2(\nu_{d,e}(\lambda)-(j+g(j)))+2g(j)+2j+1 = 2\nu_{d,e}(\lambda)+1$. Now, if we define $j_{\max}$ to be the number of even parts appearing twice in λ and if the jth unshaded part of π comes from a part of color 2 in λ, $j > j_{\max}$. Moreover, note that $\nu_{d,e}(\lambda) = \overline{\nu}(\pi) + j_{\max}$, where $j_{\max}$ is the number of even parts appearing twice in λ. In the language of step 2 of ϕ^{-1}, we have $n_j \leq 2\overline{\nu}(\pi)+2j_{\max}+1 < 2\overline{\nu}(\pi)+2j+1$, as desired.

Next, we show how we obtain the threshold condition in Step 2 of ϕ^{-1}. Again, using (5) and the definition of $g(j)$, we can see that, for $1 \leq m \leq \frac{\tau_j}{2}$, $|\pi_{j+g(j)+m}| \geq 2(\nu_{d,e}(\lambda)-j-g(j)-m)+1+2g(j)+2j = 2\nu_{d,e}(\lambda)-2m+1$.

Note that our bijection takes a partition $\lambda \in \mathcal{P}_B$ with ν parts, ν_o odd parts, $\nu_{u,e}$ even parts appearing exactly once, $\nu_{d,e}$ different even part sizes, and ν_1 parts of color 1 to a shaded odd Ferrers diagram, π where π has $\nu_{u,e}$ shaded rows, $2k+\nu_o(\pi)$ 1's, and a top row of size $\nu_{d,e}+\nu_1$ Therefore, we have proved that

$$\sum_{\lambda\in\mathcal{P}_B} a^{\nu_o(\lambda)+2\nu_{d,e}(\lambda)} t^{\nu_{d,e}(\lambda)+\nu_1(\lambda)} z^{\nu_{u,e}(\lambda)} q^{|\lambda|} = \sum_{\pi\in\mathcal{SF}} a^{\mathrm{top}(\pi)+\nu(\pi)-1} t^{\mathrm{top}(\pi)} z^{\overline{\nu}(\pi)} q^{|\overline{\pi}|}.$$

□

Note that if we set $a=t=1$ and $z=q$ in Theorem 3.1 and multiply both sides by q, we obtain (1). Furthermore, setting $a=t=z=1$ in Theorem 3.1 results in (2).

4. Combinatorial interpretations of other mock theta functions

In this section, we explore special cases of Theorem 3.1. We begin with a combinatorial interpretation of (1). Let $\mathcal{P}_{A,k}$ be the set of 3-colored partitions into odd parts, where the largest part is $2k+1$, all parts of color 3 and size $< 2k+1$ appear once or twice, and the part $(2k+1)_3$ appears exactly once. Let $\mathcal{B}$ be a set that generalizes odd Ferrers diagrams by taking shaded odd Ferrers diagrams, replacing the zero by a 1 and replacing the 1 in any shaded row with a 2.

Then, we can rewrite (1) as

$$\sum_{k\geq 0}\sum_{\lambda\in\mathcal{P}_{A,k}} q^{|\lambda|} = \sum_{\pi\in\mathcal{B}_2} q^{|\pi|}. \tag{6}$$

Note that, for $\lambda \in \mathcal{P}_{A,k}$, subtracting 1 from every part of color 3 and changing it to color 2 gives a partition $\lambda' \in \mathcal{P}_{B,k}$. Additionally, for $\pi \in \mathcal{B}$, subtracting 1 from every even part and replacing the 1 in the top left corner with a 0 results in a shaded odd Ferrers diagram $\pi' \in \mathcal{B}$. Therefore, the bijection described in Section 2 also proves (6).

Next, we consider two third-order mock theta functions that appear as special cases of Theorem 3.1. As mentioned in [10], if we let $a=t=1$ and let $z\to 0$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_{n+1}}. \tag{7}$$

Note that this function is Watson's third-order mock theta function $\omega(q)$.

If, instead, we let $t=1$, $a=-q$, and $z\to 0$, we obtain

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+2)}}{(-q^2;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{(-q^2;q^2)_{n+1}}.$$

If we multiply both sides by q^2 then take $q\to q^{1/2}$, we get

$$\sum_{n=1}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n}{(-q;q)_n}. \tag{8}$$

Note that the left-hand side of (8) can be expressed in terms of Ramanujan's third-order mock theta function

$$f(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}.$$

Using these identities and the ideas presented in Section 3, we can obtain combinatorial interpretations of the coefficients of $\omega(q)$ and $f(q)$. From (7), we obtain the following interpretations for the coefficients of $\omega(q)$:

- The number of partitions where there are no skipped even parts, all even parts appear twice, and all odd parts are two colored.
- The number of odd Ferrers diagrams.

The second interpretation can also be found in [1]. Similarly, from (8), we note the following interpretations for the coefficients of $f(q)$:

- The number of two-colored partitions into odd parts, where all part sizes appear at least once with color 2, counted with weight $(-1)^{\#\text{ parts}-\#\text{ distinct part sizes}}$.
- The number of partitions counted with weight $(-1)^{\text{largest part}-\#\text{ parts}}$.

A common interpretation of the left side of (8) is that it counts the number of partitions with even rank minus the number of partitions with odd rank, where the rank of a partition (as defined by Dyson in [8]) is the size of the largest part minus the number of parts. A fascinating result of our combinatorial interpretation is that, when traced through to this identity for $f(q)$, we get that the right side of (8) can also be interpreted as counting the number of partitions with even rank minus those with odd rank.

5. A combinatorial hypergeometric series transformation

We begin this section by noting that, in our proof of Theorem 3.1, we equated the coefficients of τ^0 on both sides of the identity

$$\sum_{n=0}^{\infty}\frac{(-a\tau q;q^2)_n(-azq;q^2)_n}{(aq;q^2)_{n+1}(a^2t\tau z;q^2)_{n+1}}a^nt^nq^n=\sum_{n=0}^{\infty}\frac{(-q^2/\tau;q^2)_n(-q^2/z;q^2)_n}{(aq;q^2)_{n+1}(atq;q^2)_{n+1}}a^{2n}t^n\tau^nz^n. \tag{9}$$

We can extend our combinatorial interpretation of Theorem 3.1 to give a new combinatorial interpretation of (9). Let $\mathcal{T}_1$ be the set of triples $(k,\overline{\pi}_1,\pi_2)$ where $k\in\mathbb{Z}_{\geq 0}$, π_1 is an overpartition into odd parts of size $\leq 2k+1$ with all overlined parts $\leq 2k-1$, and π_2 is a partition into parts of size $\leq 2k$ with all odd parts distinct and parts of size 0 allowed. Let $\mathcal{T}_2$ be the set of triples (m,λ_1,λ_2) where $m\in\mathbb{Z}_{\geq 0}$, λ_1 is a 2-colored partition into even parts of size $\leq 2m$ such that parts within each color are distinct, and λ_2 is a 2-colored partition into odd parts of size $\leq 2m+1$. Then (9) is equivalent to the following combinatorial identity.

Theorem 5.1.

$$\sum_{(k,\overline{\pi}_1,\pi_2)\in\mathcal{T}_1} a^{k+\nu(\overline{\pi}_1)+2\nu_e(\pi_2)+\nu_o(\pi_2)} t^{k+\nu_e(\pi_2)} z^{\overline{\nu}(\overline{\pi}_1)+\nu_e(\pi_2)} q^{k+|\overline{\pi}_1|+|\pi_2|} \tau^{\nu(\pi_2)}$$
$$= \sum_{(m,\lambda_1,\lambda_2)\in\mathcal{T}_2} a^{2m+\nu(\lambda_2)} t^{m+\nu_1(\lambda_2)} z^{m-\nu_2(\lambda_1)} \tau^{m-\nu_1(\lambda_1)} q^{m+|\lambda_1|+|\lambda_2|}$$

A direct bijective proof of this interpretation would be a welcome addition to the q-hypergeometric series literature.

References

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